The complexity of positive equality-free first-order logic II: the four-element case

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We are interested in a parameterisation of the model checking problem by the model. Fix a logic $\mathcal{L}$ and fix $B$. The problem “$\mathcal{L}(B)$” has

- Input: a sentence $\varphi$ of $\mathcal{L}$.
- Question: does $B \models \varphi$?
We are interested in a parameterisation of the model checking problem by the model. Fix a logic $\mathcal{L}$ and fix $B$. The problem “$\mathcal{L}(B)$” has

- Input: a sentence $\varphi$ of $\mathcal{L}$.
- Question: does $B \models \varphi$?

We consider fragments $\mathcal{L}$ of FO and structures $B$ that are relational and finite.
Consider the twin fragments

\{\exists, \land\}-\text{FO} \quad \text{and} \quad \{\exists, \land, =\}-\text{FO}.

These are the same fragment,\(^1\) since equality may be propagated out by substitution. E.g.,

\[ \exists \nu_1, \nu_2, \nu_3 \; E(\nu_1, \nu_3) \land \nu_1 = \nu_2 \land E(\nu_3, \nu_2) \]

becomes

\[ \exists \nu_1, \nu_3 \; E(\nu_1, \nu_3) \land E(\nu_3, \nu_1). \]

\(^1\)Except, possibly, for degenerate sentences involving only equalities, which are equivalent to \(\exists \nu_1 \exists \nu_2 \; \nu_1 = \nu_2.\)
The problem $\{\exists, \land\}$-FO($B$), with
- Input: a sentence $\varphi$ of $\{\exists, \land\}$-FO.
- Question: does $B \models \varphi$?

is better known as the ("non-uniform") constraint satisfaction problem CSP($B$).
The question as to which complexities this may attain has attracted much attention. A dichotomy – between P and NP-complete – is conjectured.
Now consider the fragments

$\{\exists, \forall, \land\}$-FO and $\{\exists, \forall, \land, =\}$-FO.

Again these fragments (almost) coincide, as equalities may be propagated out.$^2$ The problem $\{\exists, \forall, \land\}$-FO$(B)$, with

- Input: a sentence $\varphi$ of $\{\forall, \exists, \land\}$-FO.
- Question: does $B \models \varphi$?

is better known as the ("non-uniform") quantified constraint satisfaction problem QCSP$(B)$.

For these problems, a trichotomy – between P, NP-complete and Pspace-complete – should be conjectured.

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$^2$Except for degenerate sentences equivalent to $\forall v_1 \forall v_2 \ v_1 = v_2$, $\exists v_1 \forall v_2 \ v_1 = v_2$ or (possibly) $\exists v_1 \exists v_2 \ v_1 = v_2$. 
There are many other fragments of \( \text{FO} \), given by subsets of

\[
\{\neg, \exists, \forall, \land, \lor, =, \neq\},
\]

that are ripe for consideration.

It has long been known that the classes with negation,

- \( \{\neg, \exists, \forall, \land, \lor, =\}-\text{FO}(B) \) and
- \( \{\neg, \exists, \forall, \land, \lor\}-\text{FO}(B) \),

exhibit dichotomy between \( \text{P} \) and \( \text{Pspace-complete} \).

For the former, the criterion for hardness is \( ||B|| > 1 \); for the latter, it is \( B \)'s containing a relation that is neither empty nor contains all tuples.
Some classes, such as \{∃, ∨, =\}-FO(B), are readily seen to only have problems in L.
Other classes, such as \{∃, ∧, ∨\}-FO(B) and \{∃, ∧, ∨, =\}-FO(B), exhibit near trivial dichotomy: here between L and NP-complete, with the criterion for easiness being the presence of an \(x \in B\) s.t. all non-empty relations of \(B\) are \(x\)-valid (have a self-loop at \(x\)).
Further, some classes inter-reduce in complexity. For $B$, define $\overline{B}$ over the same universe as $B$ but with relations $R^{\overline{B}}$ which are the set-theoretic complements of $R^{B}$.

By de Morgan’s laws, there is an obvious polynomial equivalence between (the complement of the problem) $\{\exists, \land\}$-$\text{FO}(B)$ and $\{\forall, \lor\}$-$\text{FO}(\overline{B})$.

It follows that $\{\exists, \land\}$-$\text{FO}(B)$, i.e. $\text{CSP}(B)$, exhibits dichotomy between P and NP-complete iff $\{\forall, \lor\}$-$\text{FO}(B)$ exhibits dichotomy between P and co-NP-complete.
Through these inter-reductions we can see that the classifications for yet more classes result simply from the Schaefer cases for boolean CSP and QCSP. This is because computational hardness is trivial for $||B|| \geq 3$. This applies to each of:

- $\{\exists, \land, \neq\}$-FO($B$)
- $\{\forall, \lor, =\}$-FO($B$)
- $\{\forall, \exists, \land, =\}$-FO($B$)
- $\{\forall, \exists, \lor, \neq\}$-FO($B$)

In all cases the classes display dichotomy.
The class \( \{\forall, \exists, \&, \|\} - \text{FO}(B) \) can be shown to exhibit a trivial dichotomy between P and Pspace-complete, the criterion for hardness again being \( \|B\| > 1 \).
We are quickly left with only the class \( \{\forall, \exists, \&, \|\} - \text{FO}(B) \). Note that there is no obvious way to propagate out instances of equality in positive FO, \( \{\forall, \exists, \&, \|\} - \text{FO} \).
The problem $\{\exists, \forall, \land, \lor\}$-FO($B$) has

- Input: a sentence $\varphi$ of $\{\exists, \forall, \land, \lor\}$-FO, i.e. positive equality-free FO.
- Question: does $B \models \varphi$?

The hope is that:

I. This class has a non-trivial polychotomy, and that it

II. is not as difficult to find as those for the CSP and QCSP.
The algebraic method has been used to great effect in the study of both the CSP and the QCSP. The relevant connections are:

- \[ \langle B \rangle \{\exists, \land, =\} - \text{FO} = \text{Inv}(\text{Pol}(B)). \]
- \[ \langle B \rangle \{\exists, \forall, \land, =\} - \text{FO} = \text{Inv}(\text{sPol}(B)). \]

Many other connections exist, for example

- \[ \langle B \rangle \{\exists, \land, \lor, =\} - \text{FO} = \text{Inv}(\text{End}(B)). \]
- \[ \langle B \rangle \{\exists, \forall, \land, \lor, =\} - \text{FO} = \langle B \rangle \{\neg, \exists, \forall, \land, \lor, =\} - \text{FO} = \text{Inv}(\text{Aut}(B)). \]

\(^3\)On infinite \( B \) these fragments have different connections, as surjective endomorphisms are not necessarily automorphisms.
The significance of these connections is that they enable one to prove, if

$$\text{Pol}(B) \subseteq \text{Pol}(B')$$

then

$$\{\exists, \land, =\}-\text{FO}(B') \leq_L \{\exists, \land, =\}-\text{FO}(B).$$

In the absence of equality, the connections become more complicated – with operations replaced by hyper-operations.

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4 This makes sense only if $B$ and $B'$ share a common domain.
A *surjective hyper-operation* (shop) on a set $S$ is a function

$$f : S \to \mathcal{P}(S) \setminus \{\emptyset\}$$

such that, for all $y \in S$, there exists $x \in S$ s.t. $y \in f(x)$.

A *surjective hyper-endomorphism* (she) of $B$ is a surjective hyper-operation $f$ on (the domain of) $B$ that satisfies, for all extensional relations $R$ of $B$,

- if $R(x_1, \ldots, x_i) \in B$ then, for all $y_1 \in f(x_1), \ldots, y_i \in f(x_i)$, $R(y_1, \ldots, y_i) \in B$. 
Example. Shes of interest.$^5$

Note the convention for marking-up on the right-hand example. We drop the curly brackets and the commas.
Let $\text{shE}(B)$ be the class of shes of a structure $B$. If $F$ is a set of shops then $\text{Inv}(F)$ is the set of relations of which $F$ are shes. We have the relationship

$$\langle B \rangle \{\exists, \forall, \wedge, \vee\}-\text{FO} = \text{Inv}(\text{shE}(B)),$$

and its consequence

$$\text{shE}(B) \subseteq \text{shE}(B') \text{ implies } \{\exists, \forall, \wedge, \vee\}-\text{FO}(B') \leq_L \{\exists, \forall, \wedge, \vee\}-\text{FO}(B),$$

which will assist us in classifying the complexities of the problem $\{\exists, \forall, \wedge, \vee\}-\text{FO}(B)$. 
The *identity* shop \( id_S \) is defined by \( x \mapsto \{ x \} \). Given shops \( f \) and \( g \), define the *composition* \( g \circ f \) by

\[
x \mapsto \{ z : \exists y \ z \in g(y) \land y \in f(x) \}.
\]

Finally, a shop \( f \) is a *sub-shop* of \( g \) – denoted \( f \subseteq g \) – if \( f(x) \subseteq g(x) \), for all \( x \).

A set of shops on \( S \) is a *down-shop-monoid*, if it contains \( id_S \), and is closed under composition and sub-shops.\(^6\)

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\(^6\) Not all sub-shops of a shop are surjective – we are only concerned with those that are.
id_B is a she of all structures, and, if \( f \) and \( g \) are shes of \( B \), then so is \( g \circ f \). Further, if \( g \) is a she of \( B \), then so is \( f \) for all \( f \subseteq g \). It follows that \( \text{shE}(B) \) is always a down-shop-monoid. The down-shop-monoids of \( B \) form a lattice under (set-theoretic) inclusion and the sets of relations closed under \( \{\exists, \forall, \wedge, \vee\}\)-FO definability form another such lattice. These lattices are isomorphic via the operators Inv and shE, which form a Galois connection.
Example. Shes of interest.\(^7\)

\[
\begin{aligned}
\begin{array}{cccc}
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & 1 & 2 \\
\end{array}
\end{aligned}
\]

\[
\begin{aligned}
\text{list of shes}
\end{aligned}
\]

\[
\begin{aligned}
\text{:= } \langle \begin{array}{c}
0 \\
1 \\
2 \\
0 \\
1 \\
2 \\
\end{array} \rangle
& \text{ and }
\langle \begin{array}{c}
0 \\
1 \\
2 \\
0 \\
1 \\
2 \\
\end{array} \rangle
\end{aligned}
\]

\(^7\)We use \( \langle f \rangle \) to specify the down-shop-monoid generated by \( f \).
Let $S$ be a finite set with distinct elements $c, d$. We define the following types of shops.

$$A_c(x) := \begin{cases} S & \text{if } x = c \\ ? & \text{otherwise.} \end{cases}$$

$$E_c(x) := \{?, c\}$$

$$\forall\exists_{c,d}(x) := \begin{cases} S & \text{if } x = c \\ \{d\} & \text{otherwise.} \end{cases}$$
Let $S$ be a finite set with distinct elements $c, d$. We define the following types of shops.

$$A_c(x) := \begin{cases} S & \text{if } x = c \\ \{?\} & \text{otherwise.} \end{cases}$$

\[
\begin{array}{c|ccc}
0 & 0 \\
1 & 3 \\
2 & 0 & 1 & 2 & 3 \\
\end{array} \\
\text{e.g.}
\]

$$E_c(x) := \{?, c\}$$

\[
\begin{array}{c|ccc}
0 & 0 & 12 \\
1 & 1 \\
2 & 12 \\
3 & 13 \\
\end{array} \\
\text{e.g.}
\]

$$\forall \exists_{c,d}(x) := \begin{cases} S & \text{if } x = c \\ \{d\} & \text{otherwise.} \end{cases}$$

\[
\begin{array}{c|ccc}
0 & 0 & 12 & 3 \\
1 & 1 & 2 \\
2 & 2 \\
3 & 2 \\
\end{array} \\
\text{e.g.}
\]
If $B$ has a she of the form on the left, then $\{\exists, \forall, \land, \lor\}$-FO($B$) is in the complexity class in the middle (broadly) for the reason on the right.

$A_c \in$ NP evaluate all $\forall$ vars $c$.
$E_c \in$ co-NP evaluate all $\exists$ vars $c$.
$\forall\exists_{c,d} \in$ L simultaneously evaluate all $\forall$ vars $c$ and $\exists$ vars $d$. 
We consider the case where $|B| := \{0, 1\}$. There are five down-shop-monoids in this case.

**Theorem (Madelaine-M. 2009)**

If $\text{shE}(B)$ is green above, then $\{\exists, \forall, \land, \lor\}$-$\text{FO}(B)$ is in L; otherwise it is Pspace-complete.
When $|B| := \{0, 1, 2\}$, there are rather more.
However, most of those correspond to green “L” cases. The rest:

Theorem (Madelaine-M. 2009)

If $\text{shE}(B)$ is green, blue or red, above, then $\{\exists, \forall, \land, \lor\}$-$\text{FO}(B)$ is in $L$, is $\text{NP}$-complete or is co-$\text{NP}$-complete, respectively; otherwise it is $\text{Pspace}$-complete.
Conjecture

- If shE(B) contains an $\forall\exists$-shop then $\{\exists, \forall, \wedge, \vee\}$-FO($B$) is in $L$.
- If shE(B) contains no $\forall\exists$-shop but contains an $A$-shop then $\{\exists, \forall, \wedge, \vee\}$-FO($B$) is NP-complete.
- If shE(B) contains no $\forall\exists$-shop but contains an $E$-shop then $\{\exists, \forall, \wedge, \vee\}$-FO($B$) is co-NP-complete.
- Otherwise, $\{\exists, \forall, \wedge, \vee\}$-FO($B$) is Pspace-complete.

The conjecture is verified in the cases $||B|| = 2, 3$. 
Conjecture

- If sh\(E(B)\) contains an \(\forall \exists\)-shop then \(\{\exists, \forall, \land, \lor\}\)-FO\((B)\) is in \(L\).
- If sh\(E(B)\) contains no \(\forall \exists\)-shop but contains an A-shop then \(\{\exists, \forall, \land, \lor\}\)-FO\((B)\) is \(\text{NP-complete}\).
- If sh\(E(B)\) contains no \(\forall \exists\)-shop but contains an E-shop then \(\{\exists, \forall, \land, \lor\}\)-FO\((B)\) is \(\text{co-NP-complete}\).
- Otherwise, \(\{\exists, \forall, \land, \lor\}\)-FO\((B)\) is \(\text{Pspace-complete}\).

The conjecture is verified in the cases \(||B|| = 2, 3|.|\)

**Theorem (CSL 2010)**

*The conjecture is true in the case \(||B|| = 4|.|*
Contributions. Why should anyone care that the four-element case is solved?

• The four-element case throws up monoid classes unlike any that appear at the three-element level.

• The verification of the maximal monoid classes, done by hand in the three-element case, is undertaken by computer in the four-element case. Still the search space is too large to simply cover – instead the computer verifies the inductive step of a mathematical induction.

The key is in finding all of the maximal monoids in each complexity class.
Example. Maximal Pspace-complete monoids. The four maximal Pspace-complete monoids for $|B| = 3$ are drawn boxed below (in two classes).

There are corresponding maximal NP-complete monoids and co-NP-complete monoids.
Example. The twenty maximal Pspace-complete monoids for $\|B\| = 4$ are listed below (in five classes).

<table>
<thead>
<tr>
<th>Class I</th>
<th>Class II</th>
<th>Class III</th>
<th>Class IV</th>
<th>Class V</th>
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The hard part is in proving there are no others.
Let $M$ be a maximal Pspace-complete monoid. We prove by induction that

- if $M$ contains some $k$ elements other than the identity, then these $k$ elements must all be within one of $M_1, \ldots, M_{20}$.

The computer verifies the inductive step.
Conjecture (Theorem Madelaine-M. 2010+?)

- If \( \text{shE}(B) \) contains an \( \forall \exists \)-shop then \( \{\exists, \forall, \land, \lor\}\text{-FO}(B) \) is in \( L \).
- If \( \text{shE}(B) \) contains no \( \forall \exists \)-shop but contains an \( A \)-shop then \( \{\exists, \forall, \land, \lor\}\text{-FO}(B) \) is \( \text{NP-complete} \).
- If \( \text{shE}(B) \) contains no \( \forall \exists \)-shop but contains an \( E \)-shop then \( \{\exists, \forall, \land, \lor\}\text{-FO}(B) \) is \( \text{co-NP-complete} \).
- Otherwise, \( \{\exists, \forall, \land, \lor\}\text{-FO}(B) \) is \( \text{Pspace-complete} \).